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The diffusive interface in double-diffusive convection

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A model of the diffusive interface in double-diffusive convection at high Rayleigh number is proposed. The interface is assumed to have a double structure: two marginally stable boundary layers from which blobs or thermals arise on the outer edges of the interface, separated by a diffusive core across which all transport takes place by molecular diffusion. The model is time-independent and comparison is made with unsteady 'run-down' experiments on the assumption that the experiments run down through a sequence of equilibrium states each of which can be considered separately. The model predicts a constant ratio of the buoyancy fluxes of the two components at a value equal to the square root of the ratio of their molecular diffusivities, and individual fluxes in reasonable agreement with the available experimental data. Some time-dependent features of the model are also examined.

1. Introduction

Convective motions can occur in a stably stratified fluid when there are two components contributing to the density which diffuse at different rates. This convection, called double-diffusive convection, takes different forms depending on the distribution of the two components in the fluid. One component must be unstably distributed in the fluid (i.e. it must cause the density to increase upwards) to provide a source of energy for the motion whilst the other component must be stably distributed to maintain an overall stable density stratification. If the slower-diffusing component is unstably distributed, the convection takes the form of tall, thin convection cells called 'fingers'. On the other hand, if the faster-diffusing component is unstably distributed, the convection occurs in a manner more similar to thermal convection: this case is called 'diffusive'.

Theories of these convective motions have dealt primarily with the extended Bénard problem of motion between two horizontal boundaries with either the fluxes or the values of the two components specified at these two boundaries. A study of the linear stability theory for such systems has been made (Baines & Gill 1969) as well as calculations on some aspects of the finite amplitude motions (Veronis 1965, 1968; Straus 1972; Huppert & Moore 1976). A comparison of these theories with experiment is very difficult for the following reason. In order to reproduce, or even approximate, the theoretical models in the laboratory it is necessary to specify either fluxes or values of two stratifying components at the top and bottom of the working section of the fluid. In Bénard convection, when only one component is required, it is possible to use heat to stratify the fluid and it is relatively simple to maintain either constant temperatures on the boundaries or constant heat flux through the system. In the case of double-diffusive convection, however, the search for two stratifying components which can be used in this way has so far proved fruitless. Heat has often been used as one component (Turner 1965; Linden 1973). The other component has almost invariably been a solute and no method has yet been found of satisfactorily adding a solute at one boundary and removing it at the other.

The simplest experiment to set up is one in which the fluid is stratified into two uniform layers, containing different amounts of the two components, separated by an interface. Generally, the system is set up in this configuration and then allowed to 'run down' as the energy is removed from the unstably distributed component (Turner 1965; Linden 1973; Shirtcliffe 1973). It is thought that the system runs down through a series of 'quasi-equilibrium' states and that the properties of the convection are, at any given time, the same as would be observed in a steady state with the values of the flow parameters as observed at that time. Apart from the time dependence and the different initial stratifications used in the experiments, there is one other essential difference between the experiments and the theories; namely the typical values of the Rayleigh numbers of the convection are much higher than those to which the theories apply.

Consequently, it is worthwhile to try and model the experiments more closely than the Bénard-type models do. Linden (1973) has suggested a model for the 'finger' interface based on the idea of the system running down through a series of quasi-equilibrium states. Reasonable agreement was found between the theory and the experiments. Here the case of a 'diffusive' interface is examined.

A feature of the convection is that the motions in the two layers above and below the interface are driven by thermals originating at the interface, in a manner very similar to the thermals observed above a heated flat plate at high Rayleigh number. The theory of Howard (1964) for the latter situation is extended to the diffusive interface. The remainder of this paper describes this model and compares it with the relevant experiments. For further information on the general nature and properties of the convection see Turner (1973, chap. 8).

2. A steady model of the diffusive interface

The two components stratifying the fluid are denoted by T and S with molecular diffusivities K_T, and K_S, respectively. For definiteness we shall denote by S the component with the smaller diffusivity, i.e. $\tau \equiv K_S/K_T < 1$. Further, we shall let T and S represent the magnitudes of the two components in density units. Then, assuming a linear equation of state, the density ρ is given by

$$\rho = \rho_0 + T + S, \tag{2.1}$$

where ρ_0 is the density of the fluid in the absence of T and S. If the subscripts u and l refer to the upper and lower layers, respectively, a diffusive interface exists between layers where

$$T_l < T_u, \quad S_l > S_u, \quad T_l + S_l > T_u + S_u. \tag{2.2}$$

The third inequality ensures that the fluid is stably stratified.

The experiments are modelled by supposing that the system is in a steady state, with differences ΔT and ΔS in T and S maintained across the interface. That is, $\Delta T = |T_l - T_u|$ and $\Delta S = |S_l - S_u|$. A measure of the stability of the interface is given by the density ratio $R_\rho = \Delta S/\Delta T$; for the lower layer to be at least as dense as the upper layer we require $R_\rho \geqslant 1$. For simplicity, we shall suppose that the depth D of each layer is the same.

When a two-layer system with the stratification implied by (2.2) is set up in the laboratory it is observed that convective motions are produced in each layer. The source of energy for this motion is the unstable distribution of the T field. The convective motion in the two layers is driven by the release of buoyant elements from the vicinity of the density interface which separates the two layers. These buoyant elements keep the two layers fairly well mixed, so to a good approximation the horizontal averages of T and S in the two layers are independent of depth.

The essential feature of the model proposed here to describe this system is that the interface consists of three parts: a central diffusive core, across which transport of T and S takes place by molecular diffusion alone, and two identical unstable boundary layers, one above and one below the core, from which the buoyant elements arise. Good experimental evidence for a central diffusive core has been provided by Shirtcliffe (1973) from examination of a sugar–salt interface. Each boundary layer grows by diffusion. Because the T field is unstable and $K_T > K_S$, the outer edge of each boundary layer is gravitationally unstable, as the stabilizing effect of the S field is restricted to a thinner region near the core. It is envisaged that at some time $t = t_*$, say, the boundary layer breaks down and a buoyant element is released. All the buoyant fluid in the boundary layer is removed and the boundary layer, now thin, begins to grow again. This cyclic mechanism was proposed for thermal convection at high Rayleigh number by Howard (1964) and has received some experimental verification for that case from observations by Sparrow, Husar & Goldstein (1970).

The interface is symmetric about the mid-plane $z = 0$ and we shall consider only the upper half of the system $0 \leqslant z \leqslant D$. We shall assume that the profiles of T and S are independent of horizontal position. At time $t = 0$ we suppose that the boundary-layer thickness is zero and that it grows by diffusion until a buoyant element is released at $t = t_*$. The buoyant material in the boundary layer is assumed to be swept away instantaneously and the original profiles of T and S restored. These profiles are shown on figure 1. The region $0 \leqslant z \leqslant z_1$ corresponds to the diffusive core, and above this is the boundary layer. We assume that at $t = 0$, the vertical gradients of T and S in the core, T_z^d and S_z^d respectively, are constant and that there are steps $\tfrac{1}{2}\delta T$ and $\tfrac{1}{2}\delta S$ in T and S at $z = z_1$. It is then straightforward to show that diffusion until time t modifies these profiles to

$$\begin{aligned} T(z,t) = T_z^d \Bigg\{ z + \frac{1}{2}\left(\frac{\Delta T}{2T_z^d} - z\right) \operatorname{erfc}\left(\frac{z_1 - z}{2(K_T t)^{\frac{1}{2}}}\right) - \frac{1}{2}\left(\frac{\Delta T}{2T_z^d} + z\right) \operatorname{erfc}\left(\frac{z_1 + z}{2(K_T t)^{\frac{1}{2}}}\right) \\ - \left(\frac{K_T t}{\pi}\right)^{\frac{1}{2}} \left[\exp\left(\frac{-(z_1 - z)^2}{4K_T t}\right) - \exp\left(\frac{-(z_1 + z)^2}{4K_T t}\right)\right]\Bigg\} \qquad (2.3) \end{aligned}$$

and an exactly analogous expression for $S(z,t)$.

The assumption of constant gradients of T and S in the core as the initial condition is an approximation. The correct initial profiles are those which are the same in the

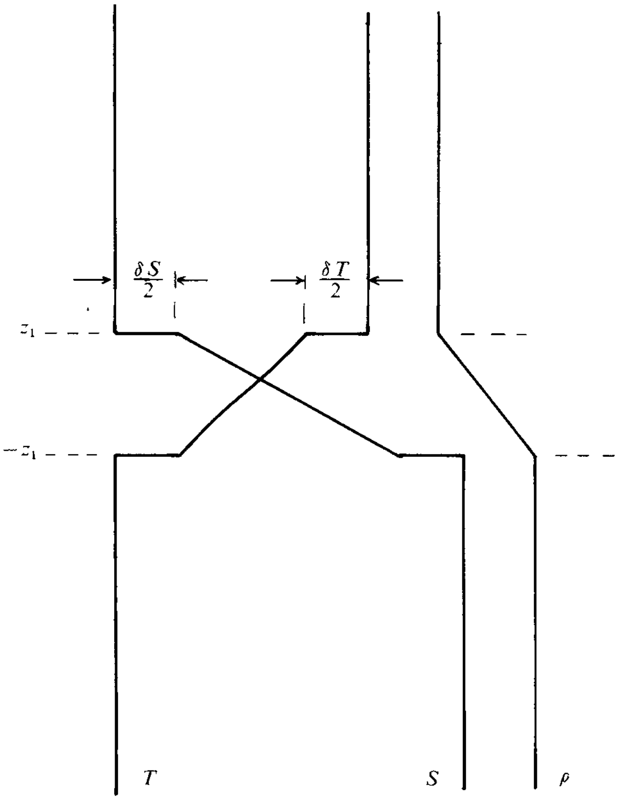

Figure 1. The idealized initial T, S and ρ profiles (in density units) of the model of the diffusive interface. The region $-z_1 \leqslant z \leqslant z_1$ is the diffusive core, in which the gradients of T and S are constant.

core at $t = t_*$ as they were at $t = 0$, since we are considering a steady state. The simplest way to determine these profiles is to assume them to be linear within $0 \leqslant z \leqslant z_1$. After diffusion has proceeded for a time t_* the profiles will have adopted shapes closer to the required repetitive form, and these may be used as the new initial condition for a second period t_*. In this way the required form would be obtained by successive approximations. Here we take only the first, linear, approximation. Provided that the interface is thick enough so that $z_1 \gg 2(K_T t_*)^{\frac{1}{2}}$, (2.3) shows the resulting error to be small except near $z = z_1$. In view of the other approximations, such as the neglect of any horizontal variations of properties, (2.3) is an adequate representation of the exact solution for our purposes. The condition that $z_1 \gg 2(K_T t_*)^{\frac{1}{2}}$ will be examined in more detail later (see §4).

The T flux F_T is given by the amount of T removed from the boundary layer per unit time. This can be written as

$$F_T = \frac{1}{t_*}\int_{z_1}^{\infty} (\tfrac{1}{2}\Delta T - T(z, t_*))\, dz.$$

Provided that $z_1 \gg 2(K_T t_*)^{\frac{1}{2}}$, a restriction we shall apply from now on, we see that

$$F_T = K_T T_z^d\{\tfrac{1}{2} + (\Delta T/2T_z^d - z_1)(\pi K_T t_*)^{-\frac{1}{2}}\}. \tag{2.4}$$

Similarly,

$$F_S = -K_S S_z^d\{\tfrac{1}{2} + (\Delta S/2S_z^d + z_1)(\pi K_S t_*)^{-\frac{1}{2}}\}. \tag{2.5}$$

The fluxes F_T^d and F_S^d through the diffusive core are given by

$$F_T^d = K_T T_z^d, \quad F_S^d = -K_S S_z^d. \tag{2.6}$$

The model presupposes that the whole system is in a steady state. As a consequence, there can be no storage or net loss of T and S from the interface, so the diffusive fluxes (2.6) must balance the convective fluxes (2.4) and (2.5). This implies that

$$\left.\begin{aligned} \Delta T/2T_z^d - z_1 &= \tfrac{1}{2}(\pi K_T t_*)^{\frac{1}{2}}, \\ \Delta S/2S_z^d + z_1 &= \tfrac{1}{2}(\pi K_S t_*)^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.7}$$

We need now to specify the position $z = z_1$ of the outer edge of the diffusive core. To do this it is assumed that all and only the fluid with density less than that in the mixed layer actively participates in the convection and is excluded from the diffusive core. Thus immediately after the release of a buoyant blob, the net density step at the edge of the core is zero as shown on figure 1. Writing this condition in terms of the steps in T and S at the edges of the core at $t = 0$, we have

$$\delta T = \delta S. \tag{2.8}$$

One important result follows immediately from this assumption. From (2.4)–(2.6) we see that the ratio of the mass fluxes

$$F_S/F_T = \tau^{\frac{1}{2}}, \tag{2.9}$$

which is constant and independent of the density ratio $R_\rho = \Delta S/\Delta T$. Then (2.6) implies that

$$T_z^d/S_z^d = -\tau^{\frac{1}{2}}. \tag{2.10}$$

Further

$$z_1 = (\Delta T - \delta T)/2T_z^d = -(\Delta S - \delta S)/2S_z^d, \tag{2.11}$$

and use of (2.8) and (2.10) gives the result

$$\delta T/\Delta T = (1-\tau^{\frac{1}{2}}R_\rho)/(1-\tau^{\frac{1}{2}}). \tag{2.12}$$

In order to determine the individual fluxes of T and S it is necessary to establish a criterion for the breakdown of the boundary layer. This is a complicated problem as, if the upper layer is of infinite depth, the boundary layer is always unstable. Consequently, we need to determine the stage at which the instabilities have grown to such a size that a breakdown of the boundary layer and the release of buoyant elements occur. At the large Rayleigh numbers ($\geqslant O(10^8)$) at which the experiments have been carried out the convection is intermittent and the release of buoyant elements is reminiscent of the collective instabilities observed in high Rayleigh number thermal convection (Busse & Whitehead 1974). A plan view of a diffusive interface as revealed by a shadowgraph is shown on figure 2 (plate 1). Similar collective instabilities are clearly visible.

Foster (1971) has made numerical calculations of the thermal convection produced at high Rayleigh numbers beneath a cooled surface. His results confirm, in general terms, the picture presented by Howard (1964) of a periodic breakdown of the boundary layer with the release of buoyant elements and its subsequent redevelopment by diffusion. We shall apply this model here in order to get estimates of the fluxes.

We define a time-dependent Rayleigh number $R = g\delta\rho d_T^3/\rho_0 \nu K_T$ for the boundary layer and assume that the buoyant element is released when R is greater than some critical value $R_c = O(10^3)$. Here $\delta\rho$ and d_T are appropriate measures, respectively, of the density anomaly and the thickness of the boundary layer. We have used the T diffusivity in R as this is the higher of the two diffusivities and because T is the destabilizing agent.

The T Nusselt number N_T is given by

$$N_T = 2T_z^d D/\Delta T, \tag{2.13}$$

as the T flux into the upper layer is given by F_T^d [see (2.6)]. Use of (2.7) and (2.11) implies that

$$\delta T = T_z^d(\pi K_T t_*)^{\frac{1}{2}}. \tag{2.14}$$

Further, we shall specify $\delta\rho$ more closely by writing

$$\delta\rho d_T = \int_{z_1}^{\infty} \left(\frac{\Delta T}{2} - \frac{\Delta S}{2} - T(z, t_*) + S(z, t_*)\right) dz,$$

and from (2.3) and the equivalent expression for $S(z, t)$ we then get

$$\delta\rho d_T = \delta T(1-\tau^{\frac{1}{2}})\,(K_T t_*/\pi)^{\frac{1}{2}}. \tag{2.15}$$

Finally, we need an estimate for d_T, the boundary-layer thickness. There are two boundary-layer length scales associated with the different diffusivities of the two components. As the S boundary layer is stable, the position of the maximum density inversion does not occur at $z = z_1$, but is displaced outwards by an amount proportional to $(K_S t_*)^{\frac{1}{2}}$. A realistic scale for the combined density profile is

$$d_T = (\pi K_T t_*)^{\frac{1}{2}} - (\pi K_S t_*)^{\frac{1}{2}} = (\pi K_T t_*)^{\frac{1}{2}}\,(1-\tau^{\frac{1}{2}}). \tag{2.16}$$

In the limit $\tau \to 0$, when the diffusion of the S field is unimportant, $d_T = (\pi K_T t_*)^{\frac{1}{2}}$, which is the appropriate scale length for the diffusion of the T boundary layer. When $\tau \to 1$, $d_T \to 0$, reflecting the fact that the unstable part of the boundary layer is much thinner than the individual boundary layers of S and T.

Substituting for T_z^d from (2.14) and using (2.16), we find

$$N_T = \frac{2\delta T}{\Delta T}\frac{D}{d_T}(1-\tau^{\frac{1}{2}}). \tag{2.17}$$

If we define the Rayleigh number for the whole upper layer as

$$R_L = g\Delta T D^3/2\rho_0 \nu K_T, \tag{2.18}$$

we see that

$$\frac{D}{d_T} = \left(\frac{2}{\pi}\frac{R_L}{R_c}\frac{\delta T}{\Delta T}\right)^{\frac{1}{3}}. \tag{2.19}$$

From (2.12), (2.17) and (2.19) we find

$$N_T = \left(\frac{16R_L}{\pi R_c}\right)^{\frac{1}{3}} \frac{(1-\tau^{\frac{1}{2}}R_\rho)^{\frac{4}{3}}}{(1-\tau^{\frac{1}{2}})^{\frac{1}{3}}}, \tag{2.20}$$

and the T flux is given by

$$F_T = \frac{1}{(\pi R_c)^{\frac{1}{3}}} K_T \left(\frac{g}{\rho_0 \nu K_T}\right)^{\frac{1}{3}} \frac{(1-\tau^{\frac{1}{2}}R_\rho)^{\frac{4}{3}}}{(1-\tau^{\frac{1}{2}})^{\frac{1}{3}}} (\Delta T)^{\frac{4}{3}}. \tag{2.21}$$

The thickness of the diffusive interface can also be determined from the above relations. This is a rather imprecisely defined quantity so we shall use a form favoured by experimentalists (e.g. Crapper & Linden 1974; Marmorino & Caldwell 1976), namely

$$h_T = \Delta T / T_z^d. \tag{2.22}$$

Equating the fluxes F_T^d and F_T, we get from (2.6) and (2.21)

$$h_T = (\pi R_c)^{\frac{1}{3}} \left(\frac{\rho_0 \nu K_T}{g}\right)^{\frac{1}{3}} \frac{(1-\tau^{\frac{1}{2}})^{\frac{1}{3}}}{(1-\tau^{\frac{1}{2}} R_\rho)^{\frac{4}{3}}} (\Delta T)^{-\frac{1}{3}}. \tag{2.23}$$

For the sake of making explicit comparison with Marmorino & Caldwell (1976) we shall rewrite (2.23) with the aid of (2.22) in the form

$$h_T = (\pi R_c)^{\frac{1}{4}} K_T^{\frac{1}{4}} \left(\frac{\rho_0 \nu K_T}{g}\right)^{\frac{1}{4}} \frac{(1-\tau^{\frac{1}{2}})^{\frac{1}{4}}}{1-\tau^{\frac{1}{2}} R_\rho} F_T^{-\frac{1}{4}}. \tag{2.24}$$

3. Time-dependent effects

It was mentioned in the introduction that all the experiments carried out on a diffusive system have been 'run-down' situations. It is of interest, therefore, to consider the effects of time dependence on the model described above. A combination of the results obtained in § 2 for the steady state and those derived below should then provide a description of a run-down experiment as the system passes through a series of quasi-equilibrium states.

Suppose that all boundaries of the system are impervious to both T and S. Then the flux across the interface produces a change in ΔT (and ΔS) with time given by

$$F_T = -\tfrac{1}{2} D d(\Delta T)/dt. \tag{3.1}$$

As $F_T = F_T^d$, (2.6) and (2.22) give

$$\tfrac{1}{2} D d(\Delta T)/dt = -K_T \Delta T / h_T, \tag{3.2}$$

i.e.

$$h_T \propto t.$$

Hence there is a linear growth of the interface thickness with time. Furthermore, (2.21) and (3.1) give (to a good approximation)

$$\Delta T \propto t^{-3}. \tag{3.3}$$

It is also clear that the density difference across the interface ($\Delta\rho = \Delta S - \Delta T$) increases with time (i.e. as the convection runs down). For

$$d(\Delta\rho)/dt = d(\Delta S)/dt + d(\Delta T)/dt = (2/D) F_T (1 - F_S/F_T) = (2/D) F_T (1-\tau^{\frac{1}{2}}) \tag{3.4}$$

by (2.10), thus as $\tau < 1$, $d(\Delta\rho)/dt > 0$.

If, on the other hand, the top and bottom boundaries are impermeable to S but will conduct T (say, for example, when using salt (S) and heat (T) as the two components), then (3.1) must be replaced by

$$F_S = -\tfrac{1}{2} D d(\Delta S)/dt. \tag{3.5}$$

Let F_l and F_u represent the T fluxes across the bottom and top boundaries, respectively. Then

$$Dd(\Delta T)/dt = F_l + F_u - 2F_T. \tag{3.6}$$

Writing (2.21) as

$$F_T = A(1-\tau^{\frac{1}{2}}R_\rho)^{\frac{4}{3}}(\Delta T)^{\frac{4}{3}}, \tag{3.7}$$

where $A = (\pi R_c)^{-\frac{1}{3}}\, K_T(g/\rho_0\, \nu K_T)^{\frac{1}{3}}\, (1-\tau^{\frac{1}{2}})^{-\frac{1}{3}}$, and using (2.9), we find that

$$\frac{dF_T}{dt} = \frac{4}{3}\frac{A^{\frac{3}{4}}}{D}(1-\tau)F_T^{\frac{1}{4}}(F_l + F_u - 2F_T). \tag{3.8}$$

Experiments using heat and salt have usually been set up (Turner 1965; Crapper 1975) by introducing two layers of different salinities into the tank and then applying a heat flux from below. Then (3.8) shows that the heat flow F_T will increase until

$$F_T = (F_l + F_u)/2), \tag{3.9}$$

at which point it will remain steady. Substituting (3.9) into (3.6) reveals that

$$d(\Delta T)/dt = 0,$$

although in the case where $F_u = 0$ the temperatures of both layers increase with time. From (3.8) it is possible to estimate the time t_s taken to reach the steady flux (3.9). As a lower bound we find

$$t_s \sim \frac{D}{(2A)^{\frac{3}{4}}\,(1-\tau^{\frac{1}{2}}R_\rho)\,(F_l + F_u)^{\frac{1}{4}}}.$$

For the kinds of heating rates used by both Turner and Crapper $t_s \sim O(1 \text{ hour})$.

4. Comparison with experiment

Experiments have been carried out using heat and salt (Turner 1965; Crapper 1975; Marmorino & Caldwell 1976) and sugar and salt (Shirtcliffe 1973). Turner and Crapper both considered the properties of the convection when a heat flux was applied through the bottom boundary of the tank: all other boundaries were insulated. Marmorino & Caldwell repeated some of these earlier experiments and conducted experiments where heat was also extracted from the top of the upper layer. Shirtcliffe's experiments were all 'run down' from an initial state with all boundaries non-conductive. For the heat/salt experiments $\tau = 0{\cdot}012$ and for the sugar/salt experiments $\tau = 0{\cdot}33$.

Shirtcliffe (1973) measured profiles of T and S through a sugar/salt diffusive interface. In a 'run-down' situation he found that the vertical gradients of T and S in the interior of the interface were constant, and that these gradients were large enough to support, by diffusion alone, the observed fluxes of T and S across the interface. In some cases his profiles also show density inversions at the edges of the interface, consistent with the notion of unstable boundary layers separated by a diffusive core. These observations support the idealized model of the interface as discussed in § 2 (see figure 1).

The flux ratio $R_f = F_S/F_T$ has been measured by all the above-mentioned workers For the sugar/salt case Shirtcliffe (1973) found $R_f = 0{\cdot}60 \pm 0{\cdot}03$ for $1{\cdot}1 \leqslant R_\rho \leqslant 2{\cdot}25$, which is to be compared with the predicted value of $\tau^{\frac{1}{2}} = 0{\cdot}577$. There is still substantial disagreement between the various values of R_f measured for the heat/salt system. Turner (1965) and Crapper (1975) both found a constant flux ratio

$$R_f = 0{\cdot}15 \pm 0{\cdot}02 \quad \text{for} \quad 2 \leqslant R_\rho \leqslant 7.$$

For $R_\rho \leqslant 2$, the flux ratio is a function of the density ratio R_ρ and $R_f \rightarrow 1$ as $R_\rho \rightarrow 1$. More recently, Marmorino & Caldwell (1976) measured flux ratios as high as 0·4 for heat/salt for $R_\rho \geqslant 2$ when the heat flux into the bottom of their tank was reduced by two orders of magnitude from the values used by Turner and by Crapper (they also found $R_f = 0{\cdot}15$ at equivalent heat fluxes). Even with this uncertainty in the measured values of the flux ratio, the smallest value $R_f = 0{\cdot}15$ is still significantly above that predicted by the model $\tau^{\frac{1}{2}} = 0{\cdot}11$.

The flux of T through the interface has also been measured by all the abovementioned experimenters. Following Turner (1965), the flux has been non-dimensionalized with respect to the flux F_T^{SP} through a fictitious solid conducting plane placed at the interface, where

$$F_T^{SP} = 0{\cdot}085\, K_T(g/\rho_0\, \nu K_T)^{\frac{1}{3}}. \tag{4.1}$$

Then (2.21) may be written as

$$F_T = F_T^*(R_\rho, \tau)\, F_T^{SP}, \tag{4.2}$$

where

$$F_T^* = \frac{1}{(\pi R_c)^{\frac{1}{3}}\, 0{\cdot}085} \frac{(1-\tau^{\frac{1}{2}}R_\rho)^{\frac{4}{3}}}{(1-\tau^{\frac{1}{2}})^{\frac{1}{3}}}. \tag{4.3}$$

We shall take, in line with Howard's (1964) original model, $R_c = 1629$. Then (4.3) becomes

$$F_T^* = \frac{1}{\pi^{\frac{1}{3}}} \frac{(1-\tau^{\frac{1}{2}}R_\rho)^{\frac{4}{3}}}{(1-\tau^{\frac{1}{2}})^{\frac{1}{3}}}. \tag{4.4}$$

Figure 3 shows a plot of (4.4) for heat/salt: shown for comparison are the data of Turner (1965). The value of $\tau^{\frac{1}{2}}$ is taken to be 0·11, which is appropriate for the average temperatures and salinities used by Turner in his experiments. The data of Crapper (1975) and Marmorino & Caldwell (1976) are consistent with Turner's data and are not included. There is good agreement between the model and the data in the range $3 \leqslant R_\rho \leqslant 7$, but the model underestimates the measured fluxes both as $R_\rho \rightarrow 1$ and for $R_\rho > 8$. Figure 4 shows the equivalent curve for the sugar/salt case, where $\tau^{\frac{1}{2}} = 0{\cdot}6$ plotted against the data of Shirtcliffe (1973). Again there is good agreement between the model and the data for intermediate value of the density ratio $1{\cdot}2 \leqslant R_\rho \leqslant 1{\cdot}65$, but the model underestimates the flux for values of R_ρ outside that range.

The comparison between the measurements of the T flux and that predicted by the model indicates that there are three regions to consider. For intermediate values of R_ρ the model describes the observed fluxes very well and we conclude that the model is a good representation of the diffusive interface in that range. At high values of R_ρ ($R_\rho \geqslant 8$ for heat/salt, $R_\rho \geqslant 1{\cdot}7$ for sugar/salt) the model underestimates the T flux. In fact, from (2.21) we see that the model predicts that the flux is zero for

$$R_\rho = R_\rho^c = \tau^{-\frac{1}{2}}$$

($R_\rho^c = 9{\cdot}1$ for heat/salt, $R_\rho^c = 1{\cdot}73$ for sugar/salt). For values of $R_\rho > R_\rho^c$ a steady-state diffusive interface with the double structure described in §2 cannot be maintained. It is easily seen from (2.12) that for

$$R_\rho > R_\rho^c, \quad \delta T/\Delta T < 0,$$

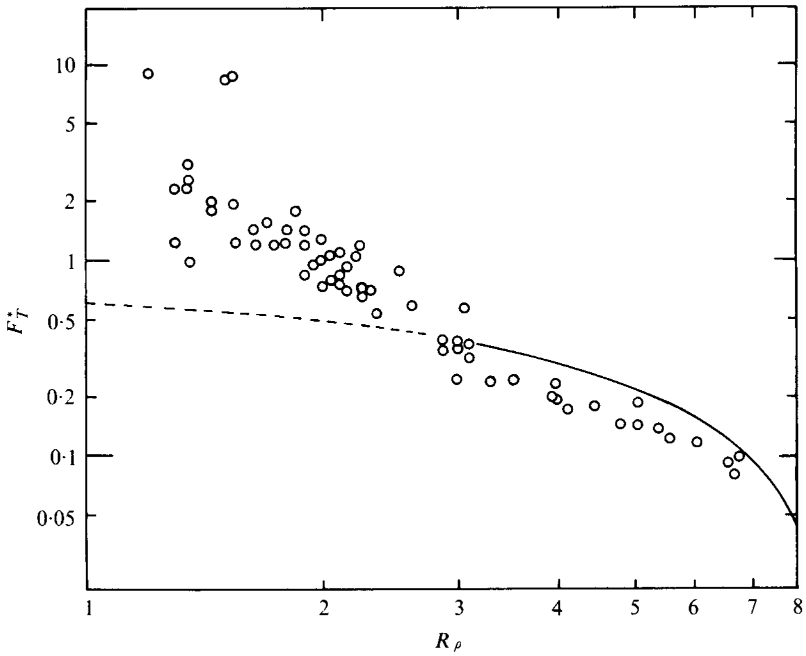

FIGURE 3. The non-dimensional T flux F_T^* for heat/salt, defined by (4.2), plotted on a log–log scale against the density ratio R_ρ. ○, Turner's (1965) data; ——, theoretical curve for heat/salt (4.4).

which implies that the step in T across the boundary layer is in the opposite sense to the step across the interface, i.e. stabilizing. There is experimental evidence that diffusive convection occurs for $R_\rho > \tau^{-\frac{1}{2}}$ (Shirtcliffe 1973), but clearly the mechanism must be different from that described in § 2. Convection driven by unstable boundary layers on either side of the interface will still work when $R_\rho > R_\rho^c$ but it is not possible for the T flux through the diffusive core to balance the (now large) S flux: in other words, condition (2.10), $T_z^d = -\tau^{\frac{1}{2}} S_z^d$, can not be satisfied.

It is of interest to consider the 'run-down' sugar/salt experiment of Shirtcliffe (1973) in view of these remarks. Details of this experiment are given in table 1 of his paper. The data were taken over a period of 90 hours, giving six discrete sets of measurements in that time. In a 'run-down' experiment both $\Delta\rho$ and R_ρ increase with time, and only the first reading (at 8·2 h) has $R_\rho < R_\rho^c$. Consequently, by following the subsequent development of the system we can see what happens as R_ρ increases beyond the value for which the present model is valid. Shirtcliffe measured the T and S gradients at the centre of the interface directly and hence the fluxes F_T^d and F_S^d. He found that, as the experiment progressed and R_ρ increased, $R_f^d = F_S^d/F_T^d$ increased from $\tau^{\frac{1}{2}}$ (at 8·2 h and 12 h) to unity. Thus, as expected, the balance described by (2.10) breaks down for $R_\rho > R_\rho^c$. Also, both F_T^d and F_S^d become greater than F_T and F_S, respectively, as R_ρ increases beyond R_ρ^c. Thus there must be a build up of T and S at the edges of diffusive core which cannot be removed by convection as there is insufficient energy in the T field. Shirtcliffe notes that $R_f = F_S/F_T$ remains constant $(= 0{\cdot}6 = \tau^{\frac{1}{2}})$ throughout. Thus it appears that the convective boundary layers still operate as described in § 2 to produce a flux ratio $R_f = \tau^{\frac{1}{2}}$ but that the diffusive core

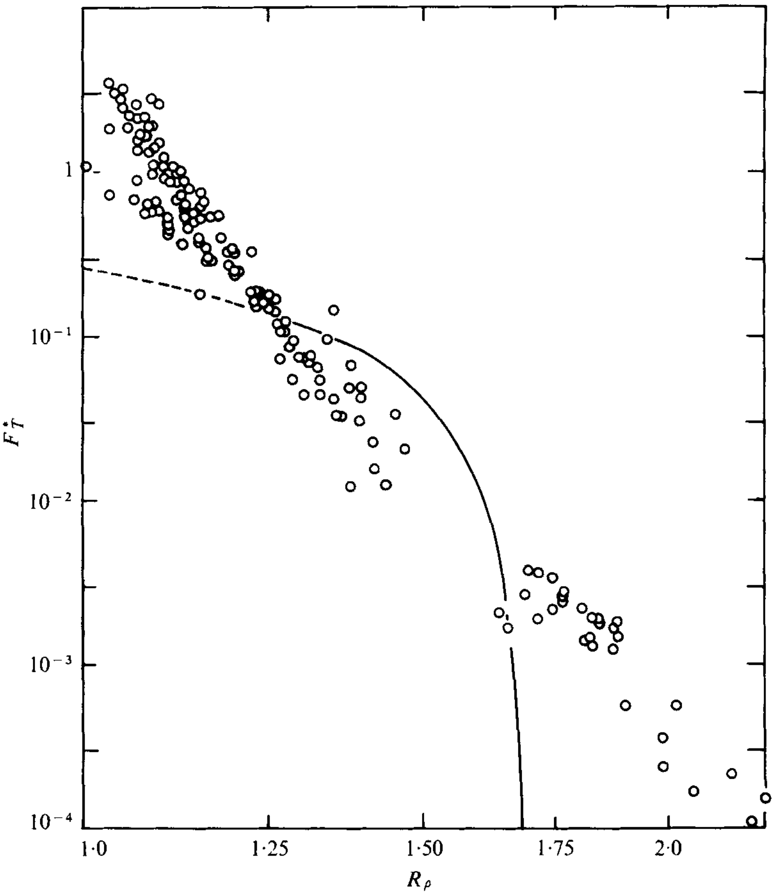

FIGURE 4. The non-dimensional T flux F_T^* for sugar/salt, defined by (4.2), plotted on a log–log scale against the density ratio R_ρ. ○, Shirtcliffe's (1973) data; ——, corresponding theoretical curve (4.5).

is no longer in balance with them. Equilibrium is lost with the imbalance of fluxes and a steady state cannot be achieved.

The third region of interest concerns the limit $R_\rho \to 1$, when the density difference across the interface becomes vanishingly small. In this limit the model underestimates the T flux for both the heat/salt and the sugar/salt interface and also does not predict the rise in the flux ratio observed in the heat/salt case. There are a number of reasons for this lack of agreement between the model and the experiments when $R_\rho \to 1$, and it is instructive to consider them in detail. First, the model, which supposes that there are constant gradients in the diffusive core, is inappropriate in this limit. From (2.12) and (2.11) we see that the thickness of the core z_1 tends to zero as $R_\rho \to 1$. Consequently, the assumption that $z_1 \gg 2(\pi K_T t_*)^{\frac{1}{2}}$ used in deriving (2.4) is not satisfied. Typical values of t_* appropriate to laboratory conditions are approximately 30 s for heat/salt and 60 s for sugar/salt. Thus we require that the heat/salt interface be at least 1 cm thick and the sugar/salt interface be 0·1 cm thick for the approximations used in deriving (2.4) to be valid. Measurements of interface thickness made by

Marmorino & Caldwell (1976) and Shirtcliffe (1973) show that these conditions are satisfied for the intermediate range of R_ρ where the agreement between the model and the data is satisfactory.

For the heat/salt interface there are other processes, not taken into account in the model, which appear to be important at low R_ρ. In all these experiments the two-layer system is set up and then driven towards $R_\rho = 1$ by heating the lower layer and (sometimes) cooling the upper layer. This imposed heat flux also generates thermals at the lower and upper boundaries of the tank which contribute to the convective motions in the layers generated by the interface itself. If these additional convective motions affect the interface then the transport through the interface may be quite different to the transport through an isolated diffusive interface. There are some indications that these convective motions may be responsible for increasing the fluxes through the interface and increasing the flux ratio as $R_\rho \to 1$. Marmorino & Caldwell (1976) found that the flux ratio was a function of the imposed heat flux for fixed R_ρ, and all the above experimenters report that the interface migrated vertically during the course of an experiment with the rate of migration increasing as $R_\rho \to 1$. Both of these observations are consistent with the idea proposed by Linden (1974) that the convective motions can entrain fluid across the interface. This effect will be largest when the stability of the interface is smallest, i.e. as $R_\rho \to 1$, and Linden (1974) showed that the increase in flux ratio as $R_\rho \to 1$ measured by Turner (1965) is consistent with this idea.

Entrainment increases as the Froude number Fr of the interface increases. Here

$$Fr = u \Big/ \left(\frac{g\Delta\rho}{\rho_0} l\right)^{\frac{1}{2}},$$

where u and l are typical velocity and length scales of the motion near the interface. For fixed R_ρ, $\Delta\rho \propto \Delta T$ and $u \propto (\Delta T)^{\frac{4}{9}}$. Consequently, $Fr \propto (\Delta T)^{-\frac{1}{18}} \propto F_T^{-\frac{1}{24}}$, and as the heat flux F_T decreases the Froude number increases, leading to an increased possibility of entrainment. Thus the larger values of R_f measured by Marmorino & Caldwell (1976) compared with those measured by Turner (1965) and Crapper (1975) for the same R_ρ are consistent with the idea that entrainment is producing a significant transport of heat and salt. Indeed, it seems likely that the interaction of the convective motions with the interface is responsible for the discrepancy between the predicted value $R_f = 0{\cdot}11$ and the observed value $R_f = 0{\cdot}15$.

In the case of the sugar/salt interface, calculations (Linden 1974) show that it is unlikely that entrainment across the interface is significant. This is borne out by the fact that the measured flux ratio, in this case, is in agreement with the predicted value and there is no evidence that it increases at low values of R_ρ. It is, of course, still true that when $R_\rho = 1$ and the whole system mixes, $R_f = 1$, but for the smallest value of R_ρ at which flux measurements have so far been obtained ($R_\rho = 1{\cdot}05$) no evidence of increasing R_f is found. An additional complicating feature at low R_ρ is the presence of a systematic interfacial wave motion which is coupled to the large-scale convection in the layers (Turner 1974). This causes significant horizontal variations in the thickness of the interface and may produce an increase in the flux due to a net increase in the local vertical gradients of T and S. The dynamics and effect of this wave motion are not understood and are in need of further experimental and theoretical study.

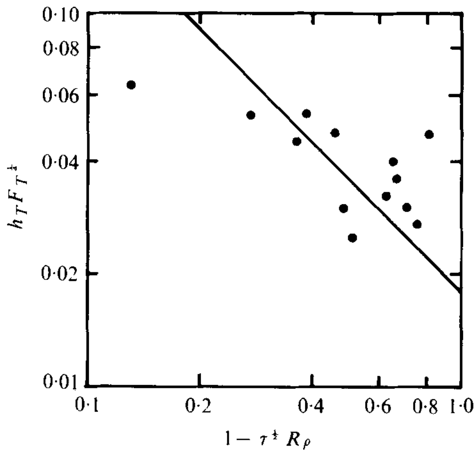

FIGURE 5. A log–log plot of $h_T F_T^{\frac{1}{3}}$ $(\mathrm{g\,cm^2/s})^{\frac{1}{3}}$ against $1-\tau^{\frac{1}{2}}R_\rho$. The data points were taken from Marmorino & Caldwell (1976) by reading off pairs of points (R_ρ, F_T) along the contours of h_T from figure 6 of their paper. The value of $\tau^{\frac{1}{2}}$ used is 0·11. The solid line is the model prediction (2.24) with values of the molecular quantities appropriate to heat and salt.

Measurements of the thickness of a heat/salt diffusive interface were made by Marmorino & Caldwell (1976). They determined a measure of the thickness from temperature–depth profiles made during their experiments, which can readily be compared with $h_T = \Delta T/T_z^d$ as defined by (2.22). The most convenient form for comparison is that of (2.24), which shows that h_T increases as F_T decreases for a fixed density ratio R_ρ, and that h_T increases as R_ρ increases for a fixed T flux F_T. These trends were also found by Marmorino & Caldwell (1976). Their data are replotted on figure 5, which shows a log–log plot of $F_T^{\frac{1}{3}} h_T$ against $1-\tau^{\frac{1}{2}}R_\rho$. The data were read at a number of points on their original figure. Also shown is a straight line representing the model prediction determined from (2.24), both the slope and the intercept given by that equation being presented without adjustment. Although the data are somewhat scattered there is good agreement between the predicted and observed interface scales.

There is little documentation of the time-dependent features of a diffusive interface. Shirtcliffe (1973) records T as a function of time as the sugar/salt system runs down. He finds that although the data fit the form $\Delta T \propto (t-t_0)^{-\gamma}$, where t_0 is a time origin, γ is in general much less than the value $\gamma = 3$ expected from (3.3). This discrepancy may be due to the fact that the validity of (3.3) depends on $1-\tau^{\frac{1}{2}}R_\rho$ being a slowly varying quantity, which is not the case in Shirtcliffe's experiments.

Information concerning the thickness of a diffusive interface as a function of time is also lacking. Some trends can be inferred from the data presented in table 1 of Shirtcliffe's (1973) paper, using the definition $h_T = \Delta T/T_z^d$ as in §2. The values of h_T determined in this way are plotted against time on figure 6. In the early stages of the experiment the growth of the interface with time is linear as described by (3.2). At later stages the growth is slower than linear but at this point the interface was an appreciable fraction of the tank depth so these values should be regarded with caution.

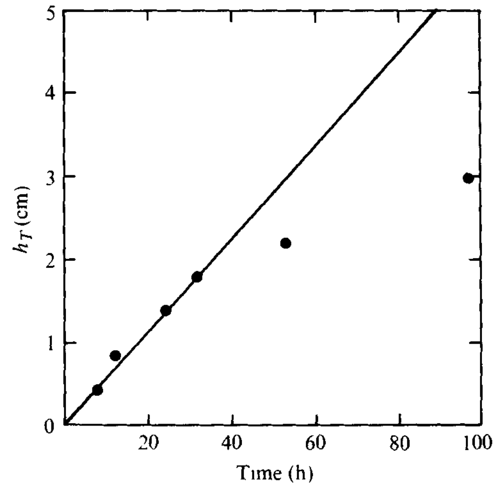

FIGURE 6. The thickness of the interface h_T(cm), taken from the data of Shirtcliffe (1973), plotted against the time t (h). Initially, the thickness is observed to increase linearly with time (as emphasized by the line drawn through the first few data points). At a later stage in the experiment the growth is slower.

5. Conclusions

In an attempt to make a direct comparison with experiments on the diffusive form of double-diffusive convection, a model of the diffusive interface at high Rayleigh number has been proposed. Recognizing the similarities between this convective flow and that above a heated horizontal boundary in a fluid at high Rayleigh number, the model of Howard (1964) for the latter situation has been extended to the two-component case. The model is based on the assumption that the interface has a diffusive core bounded above and below by unstable boundary layers from which thermal elements arise. Allied to this is the notion that a time-dependent experiment can be considered as running down through a sequence of equilibrium states, each of which may be described by a steady-state theory.

The model provides estimates of the fluxes of the two components across the interface and the thickness of the interface in terms of the difference in value of the two components in the two layers and the molecular properties of the system. It is predicted that the magnitudes of the fluxes decrease with increasing density ratio R_ρ, whilst the ratio R_f of the buoyancy fluxes is constant at a value $\tau^{\frac{1}{2}} = (K_S/K_T)^{\frac{1}{2}}$. Comparison with experiment shows that over an intermediate range of density ratios R_ρ the model provides a reasonable description of the observed transports across the interface and of the interface thickness. It is also predicted that for values of R_ρ greater than a critical value $R_\rho^c = \tau^{-\frac{1}{2}}$, which depends only on the molecular diffusivities of the two components, a steady-state interface cannot exist. One feature of this high R_ρ limit is revealed by (2.23), which shows that the interface thickness $h_T \to \infty$ as $R_\rho \to R_\rho^c$. A thickening of the interface for $R_\rho > 7$ ($R_\rho^c = 9{\cdot}1$ for heat/salt) with a tendency for an 'intermediate layer' to form was reported by Turner (1965). The form the interface takes when $R_\rho > R_\rho^c$ is not well understood and requires further investigation.

As $R_\rho \to 1$ and the density step across the interface becomes small the model underestimates the individual transports. In the heat/salt case it appears that the additional flux may be due to entrainment produced by the interaction of the convective motions with the interface. However, it seems unlikely that entrainment is important for the sugar/salt interface in the parameter range so far investigated, and the reason for the large fluxes is not known.

The possibility that entrainment may be important in certain circumstances highlights some of the difficulties in carrying out experiments of this kind. In the heat/salt experiments migration of the interface in some cases was reported by all the experimenters (Turner 1965; Crapper 1975; Marmorino & Caldwell 1976). This migration results from a net transport of fluid across the interface caused by an imbalance in the entrainment rates into the upper and lower layers. Both Turner and Crapper applied a heat flux at the bottom of the tank but had an insulating lid on the top. Consequently, the motions in the lower layer were more vigorous than those in the upper layer and the entrainment into the lower layer was larger than that into the upper layer. As a result the interface migrated upwards (Crapper 1975). On the other hand, Marmorino & Caldwell, who carried out experiments with heat removed at the top at the same rate as it was added at the bottom, also observed migration of the interface. Even in these symmetric conditions with the interface mid-way between the upper and lower boundaries, the equilibrium is unstable. If the interface is moved upwards, say, a small amount whilst the same flux is maintained through the system, the velocities in the deeper layer must increase. Hence entrainment into the lower layer will increase, continuing the upward migration of the interface. This means that at low density ratios it will be very difficult to maintain the system in a quasi-steady state.

This work was finally completed whilst we were visiting the Research School of Earth Sciences, The Australian National University. We should like to thank Professor J. S. Turner for inviting us to work there and for skilfully arranging for our visits to overlap.

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Figure 2. A shadowgraph of the plan view of a sugar/salt diffusive interface showing the collective instabilities.