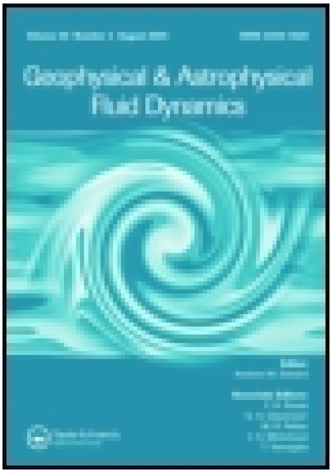

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Topographic instability and multiple equilibria on an f-plane

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Topographic Instability and Multiple Equilibria on an *f*-Plane

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The flow of a two-layer flow in a rotating channel on an f-plane over topography with sinusoidal variation of height in a direction parallel to the flow is investigated. When the two layers flow in opposite directions a resonance is found when the topographic scale matches the free mode of the system. We examine the stability of the forced mode in the vicinity of this resonance by means of a perturbation expansion of the topographic height. Both subresonant and super-resonant instabilities are found and their equilibration is examined. For small values of the dissipation multiple equilibria are found. The topographic drag releases potential energy even when the flow is baroclinically stable.

1. INTRODUCTION

In recent years a number of papers have been written on the effects of topography on flow in rotating systems. The pioneering paper was that of Charney and DeVore (1979), and this has been followed subsequently by Hart (1979), Charney and Strauss (1980), Roads (1980a, b), Davey (1980, 1981) and Pedlosky (1981). These papers have shown that for barotropic and (two-layer) baroclinic flow on a β-plane the topography can destabilize the otherwise stable flow.

There is a strong feedback mechanism associated with the instability. As the speed of the flow decreases, frictional effects become important and a wave which is out of phase with the topography is generated. This wave exerts a form drag on the topography and this, in turn, decelerates the flow still further. The system tends to an equilibrium state in which the flow is weak and the wave amplitude is large. At other values of the forcing another equilibrium state exists with a strong mean flow and small wave amplitude. The existence and stability of these multiple equilibria associated with the topographic instability have been examined in the above systems and they are found to occur when the system is in a near resonant state.

All of the above papers, with the exception of Pedlosky (1981), have employed some *ad hoc* approximations to obtain a tractable system of equations to solve. Charney and DeVore (1979) and Charney and Strauss (1980) used a highly truncated spectral model whilst Hart (1979) assumed that the topography was infinite in the meridional (cross-flow) direction. Davey (1980, 1982) has used a quasi-linear model in which advection of vorticity and self-interactions of the perturbations are neglected. It is hard to assess the effect of these approximations, particularly since many of the phenomena are associated with linear resonance. A formal approach to the problem has been made by Pedlosky (1981) which avoids these difficulties. He recognised that the fact that the earlier models had found topographic instability and the associated multiple equilibria near resonance suggests that by insisting the system be near resonance analytic progress can be made. The topography forces a free mode of the system (a Rossby wave) and the non-linear interaction between this and the mean flow can be calculated. Quasi-geostrophic flow over sinusoidal topography undulating in the direction of flow is investigated for barotropic and two-layer baroclinic flows. For the case of barotropic flow Pedlosky found that the instability displays a sensitivity to the ratio of cross-section to downstream wavenumber which was not apparent in the earlier studies. In particular, there is a possibility of sub-resonant instability.

In the case of the baroclinic flow, Pedlosky's analysis was restricted to the case where the lower layer was stationary. This implies that a larger amplitude topography is required to force the free Rossby wave and the steady-state amplitudes scale on the topographic height. The topographic instability which occurs below the threshold of the standard baroclinic instability is found to extract potential energy from the basic flow, which in turn diminishes the mean flow via the thermal wind relation. Pedlosky also showed that the presence of topography stabilized the standard mode of baroclinic instability.

In this paper I investigate the effects of topography on two-layer baroclinic flow on a f-plane. I shall adopt the approach taken by Pedlosky (1981) and investigate the near resonant response of the system. One reason for examining this system is that it is amenable to experimental verification. It is difficult to simulate two-layer baroclinic flow on a β-plane in the laboratory, but the simpler f-plane geometry is readily modelled. In addition, the present calculations are not restricted to the case of a stationary lower layer and the active lower layer implies that the system is more sensitive to the topography. In particular, if the topographic height is ($a \ll 1$) then the perturbation scales on $a^{1/3}$, a much larger amplitude. The plan of the paper is as follows. In Section 2 the basic quasi-geostrophic model is developed and the conditions for the inviscid topographic instability are derived in Section 3. The finite amplitude equilibria are examined in Section 4 and the main results are summarised in Section 5.

2. THE BAROCLINIC MODEL

Consider two-layer flow, in a rectangular channel of width L parallel to the x-axis on an f-plane, with uniform (but different) horizontal velocities in both layers. A characteristic horizontal velocity U and the channel width L are used to non-dimensionalise the variables. The quasi-geostrophic potential vorticity equations for the two layers can be written in non-dimensional form as (Pedlosky, 1979)

$$\frac{\partial}{\partial t}\{\nabla^2\psi_1 - F_1(\psi_1-\psi_2)\} + J\{\psi_1, \nabla^2\psi_1 - F_1(\psi_1-\psi_2)\}$$

$$= -r_1\nabla^2\psi_1 - r_i\nabla^2(\psi_1-\psi_2) + Q_1, \qquad (2.1a)$$

$$\frac{\partial}{\partial t}\{\nabla^2\psi_2 - F_2(\psi_2-\psi_1)\} + J\{\psi_2, \nabla^2\psi_2 - F_2(\psi_2-\psi_1) + \eta_B\}$$

$$= -r_2\nabla^2\psi_2 - r_i\nabla^2(\psi_2-\psi_1) + Q_2. \qquad (2.1b)$$

Here the subscripts 1 and 2 correspond to the upper and lower layers, respectively. The veolocity $\mathbf{U}_n$ is related to the geostrophic stream function ψ by $\mathbf{U}_n=(\partial\psi_n/\partial y, -\partial\psi_n/\partial x)$. The Froude number $F_n=f^2L^2/g'h_n$, where h_n is the depth of the layer, g' the reduced gravity between the layers and $\Omega=\frac{1}{2}f$ is the rotation rate.

The Jacobian $J(a, b)$ is defined in the usual way as

$$J(a,b)\equiv\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}. \tag{2.2}$$

The terms on the right-hand side of (2.1a, b) represent the dissipation of potential vorticity in Ekman layers at the upper and lower boundaries $(r_n\nabla^2\psi_n)$ and at the interface $[r_i\nabla^2(\psi_2-\psi_1)]$. The functions $Q_n(y)$ represent some external potential vorticity source which, in the absence of topography, would drive a steady zonal flow $U_n(y)=-\partial\psi_n/\partial y$, where

$$Q_n=r_n\frac{\partial^2\psi_n}{\partial y^2}-(-1)^n\left[r_i\left(\frac{\partial^2\psi_1}{\partial y^2}-\frac{\partial^2\psi_2}{\partial y^2}\right)\right]. \tag{2.3}$$

The bottom topography enters into the equation for the lower layer (2.1b) as the dimensionless function

$$\eta_B=\frac{h_B}{h_2U}fL, \tag{2.4}$$

where h_B is the elevation of the topography.

Write the stream function as a sum of the uniform mean flow plus the disturbance due to the topography as

$$\psi_n=U_ny+\varepsilon\phi_n. \tag{2.5}$$

Then (2.1a, b) become

$$\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right)(\nabla^2\phi_1-F_1(\phi_1-\phi_2))+F_1(U_1-U_2)\frac{\partial\phi_1}{\partial x}$$

$$+\varepsilon J(\phi_1,\nabla^2\phi_1+F_1\phi_2)=-r_1\nabla^2\phi_1-r_i\nabla^2(\phi_1-\phi_2), \tag{2.6a}$$

$$\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right)(\nabla^2\phi_2-F_2(\phi_2-\phi_1))+F_2(U_2-U_1)\frac{\partial\phi_2}{\partial x}$$

$$+\varepsilon J(\phi_2,\nabla^2\phi_2+F_2\phi_1)+J(\phi_2,\eta_B)+\frac{U_2}{\varepsilon}\frac{\partial\eta_B}{\partial x}$$

$$=-r_2\nabla^2\phi_2-r_i\nabla^2(\phi_2-\phi_1). \qquad (2.6\text{b})$$

The boundary conditions are that

$$\frac{\partial\phi_n}{\partial x}=0 \quad \text{on} \quad y=0,1, \qquad (2.7)$$

whilst any x-independent part of ϕ must satisfy

$$\frac{\partial\phi_n}{\partial y}=0 \quad \text{on} \quad y=0,1. \qquad (2.8)$$

We restrict attention to bottom topography of the form

$$\eta_B=h_0\cos kx\sin ly, \qquad (2.9)$$

where $l=n\pi$, n an integer. Resonance with the topography can occur when the left-hand sides of (2.6a, b) have *stationary*, normal mode solutions. Writing

$$U_2=-\alpha U_1, \qquad (2.10)$$

we find that stationary solutions will occur when

$$a^2=(k^2+l^2)=(\alpha^2F_1+F_2)/\alpha. \qquad (2.11)$$

Thus resonance requires $\alpha>0$ and so, relative to the topography, the two layers must be moving in opposite directions. The wave-number a_0 for the marginally stable baroclinic waves is given by $a_0^2=2(F_1F_2)^{1/2}$, and

$$a^2-a_0^2=(\alpha F_1^{1/2}-F_2^{1/2})^2/\alpha\geqq 0. \qquad (2.12)$$

Therefore, the resonant topographically forced waves have wavenumbers in excess of a_0 the baroclinic short wave cut-off, and we shall further restrict the flow by requiring $l^2 > a_0^2$ so that the system is baroclinically stable.

Following Pedlosky (1981) we insist that the flow be near resonance and write

$$U_2 = -(\alpha + \Delta)U_1,$$

where α is given by (2.11). It will become apparent that a consistent perturbation expansion is obtained with $\Delta = O(\varepsilon^2)$ and $h_0 = O(\varepsilon^3)$. The development of the topographic wave takes place on the slow time scale $T = \mu t$, where $\mu = O(\varepsilon^2)$, and we choose the dissipation $r = O(\varepsilon^2)$.

With these new scalings (2.6a, b) become

$$\left(\mu\frac{\partial}{\partial T} + U_1\frac{\partial}{\partial x}\right)(\nabla^2\phi_1 - F_1(\phi_1 - \phi_2)) + F_1U_1(1+\alpha)\frac{\partial\phi_1}{\partial x} + F_1U_1\Delta\frac{\partial\phi_1}{\partial x}$$

$$+ \varepsilon J(\phi_1, \nabla^2\phi_1 + F_1\phi_2) = -r_1\nabla^2\phi_1 - r_i\nabla^2(\phi_1 - \phi_2), \qquad (2.13a)$$

$$\left(\mu\frac{\partial}{\partial T} - \alpha U_1\frac{\partial}{\partial x} - \Delta U_1\frac{\partial}{\partial x}\right)(\nabla^2\phi_2 - F_2(\phi_2 - \phi_1)) - F_2U_1(1+\alpha)\frac{\partial\phi_2}{\partial x}$$

$$-F_2U_1\Delta\frac{\partial\phi_2}{\partial x} + \varepsilon J(\phi_2, \nabla^2\phi_2 + F_2\phi_1) + J(\phi_2, \eta_B) + \frac{U_2}{\varepsilon}\frac{\partial\eta_B}{\partial x}$$

$$= -r_2\nabla^2\phi_2 - r_i\nabla^2(\phi_2 - \phi_1). \qquad (2.13b)$$

The solution to these equations will be sought in the form of an asymptotic series,

$$\phi_n = \phi_n^{(0)} + \varepsilon\phi_n^{(1)} + \varepsilon^2\phi_n^{(2)} + \dots \qquad (2.14)$$

Substituting (2.14) into (2.13a, b) gives at the leading order

$$\nabla^2\phi_1^{(0)} + F_1(\alpha\phi_1^{(0)} + \phi_2^{(0)}) = 0,$$

$$\nabla^2\phi_2^{(0)} + F_2(\alpha^{-1}\phi_2^{(0)} + \phi_1^{(0)}) = 0, \qquad (2.15)$$

with solutions

$$\phi_n^{(0)} = A_n e^{ikx} \sin ly, \quad n=1,2, \tag{2.16}$$

where

$$A_1 = \frac{\alpha F_1}{F_2} A, \quad A_2 = A, \tag{2.17}$$

and

$$k^2 + l^2 = (\alpha^2 F_1 + F_2)/\alpha. \tag{2.18}$$

Thus the $O(1)$ problem recovers the linear, inviscid wave which resonates with the topography. The wave amplitude $A(T)$ is a slowly varying function of time whose evolution is determined by the higher order dynamics.

At $O(\varepsilon)$ the expansion procedure yields

$$U_1 \frac{\partial}{\partial x}(\nabla^2 \phi_1^{(1)} + F_1(\alpha \phi_1^{(1)} + \phi_2^{(1)}))$$

$$= -J(\phi_1^{(0)}, \nabla^2 \phi_1^{(0)} + F_1 \phi_2^{(0)}) = 0, \tag{2.19a}$$

$$-\alpha U_1 \frac{\partial}{\partial x}\left(\nabla^2 \phi_2^{(1)} + F_2\left(\frac{1}{\alpha}\phi_2^{(1)} + \phi_1^{(1)}\right)\right)$$

$$= -J(\phi_2^{(0)}, \nabla^2 \phi_2^{(0)} + F_2 \phi_1^{(0)}) = 0. \tag{2.19b}$$

Since $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are proportional, the right-hand sides of (2.19a, b) are zero. Hence the solution for $\phi_n^{(1)}$ are multiples of those for $\phi_n^{(0)}$ and can be incorporated in them, or else are x-independent corrections to the zonal flow

$$\phi_n^{(1)} = \Phi_n(y, T) \quad n=1,2. \tag{2.20}$$

Noting that $\mu=O(\varepsilon^2)$, $r=O(\varepsilon^2)$ and $h_0=O(\varepsilon^3)$, the $O(\varepsilon^2)$ problem is

$$U_1\frac{\partial}{\partial x}(\nabla^2\phi_1^{(2)}+F_1(\alpha\phi_1^{(2)}+\phi_2^{(2)}))$$

$$=-\frac{\mu}{\varepsilon^2}\frac{\partial}{\partial T}(\nabla^2\phi_1^{(0)}-F_1(\phi_1^{(0)}-\phi_2^{(0)}))-F_1U_1\frac{\Delta}{\varepsilon^2}\frac{\partial\phi_1^{(0)}}{\partial x}$$

$$-\frac{\partial\phi_1^{(0)}}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2\Phi_1}{\partial y^2}+F_1\Phi_2\right)+\frac{\partial\Phi_1}{\partial y}\frac{\partial}{\partial x}(\nabla^2\phi_1^{(0)}+F_1\phi_2^{(0)})$$

$$-\frac{r_1}{\varepsilon^2}\nabla^2\phi_1^{(0)}-\frac{r_i}{\varepsilon^2}\nabla^2(\phi_1^{(0)}-\phi_2^{(0)}), \tag{2.21a}$$

$$-\alpha U_1\frac{\partial}{\partial x}\left(\nabla^2\phi_2^{(2)}+F_2\left(\frac{\phi_2^{(2)}}{\alpha}+\phi_1^{(2)}\right)\right)$$

$$=-\frac{\mu}{\varepsilon^2}\frac{\partial}{\partial T}(\nabla^2\phi_2^{(0)}-F_2(\phi_2^{(0)}-\phi_1^{(0)}))+\frac{\Delta}{\varepsilon^2}U_1\frac{\partial}{\partial x}(\nabla^2\phi_2^{(0)}+F_2\phi_1^{(0)})$$

$$-\frac{\partial\phi_2^{(0)}}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2\Phi_2}{\partial y^2}+F_2\Phi_1\right)+\frac{\partial\Phi_2}{\partial y}\frac{\partial}{\partial x}(\nabla^2\phi_2^{(0)}+F_2\phi_1^{(0)})$$

$$+\frac{\alpha U_1}{\varepsilon^3}\frac{\partial\eta_B}{\partial x}-\frac{r_2}{\varepsilon^2}\nabla^2\phi_2^{(0)}-\frac{r_i}{\varepsilon^2}\nabla^2(\phi_2^{(0)}-\phi_1^{(0)}). \tag{2.21b}$$

In order to remove projections of the homogeneous solutions from the forcing terms in (2.21a, b) we apply the operator

$$\frac{k}{2\pi}\int_0^1 dy\int_0^{2\pi/k}e^{-ikx}\sin ly\,dx$$

to these equations. Then boundedness of $\phi_n^{(2)}$ requires that

$$\alpha I_1-\frac{1}{\alpha}I_2=0, \tag{2.22}$$

where I_1 and I_2 are the right-hand sides of (2.21a) and (2.21b), respectively, after the integration. This procedure leads to the following equation for $A(T)$:

$$\frac{\mu}{\varepsilon^2}(\alpha+1)(\alpha^4F_1^2-F_2^2)\frac{dA}{dT}-\frac{\Delta}{\varepsilon^2}ikU_1(\alpha^4F_1^2-F_2^2)A$$

$$+2Aikl\int_0^1\left[\alpha^4F_1\left(\frac{\partial^2\Phi_1}{\partial y^2}+F_1\Phi_2\right)-\alpha F_2\left(\frac{\partial^2\Phi_2}{\partial y^2}+F_2\Phi_1\right)\right.$$

$$\left.+\alpha^5F_1^2\Phi_1-F_2^2\Phi_2\right]\sin 2ly\,dy-\frac{ikU_1h_0}{\varepsilon^3}\alpha^2F_2$$

$$+\left(\frac{r_1}{\varepsilon^2}a^2\alpha^4F_1F_2-\frac{r_2}{\varepsilon^2}a^2\alpha F_2+\frac{r_i\alpha a^2}{\varepsilon^2}(1+\alpha^2)(F_2-\alpha F_1)\right)A=0. \tag{2.23}$$

In order to solve this equation for $A(T)$ it is necessary to determine Φ_1 and Φ_2. This is done by taking a zonal average of (2.13a, b), noting that $\bar{\phi}_n=\varepsilon\Phi_n$ where a bar denotes the zonal average. This gives

$$\mu\frac{\partial}{\partial T}(\nabla^2\Phi_1-F_1(\Phi_1-\Phi_2))+r_1\frac{\partial^2\Phi_1}{\partial y^2}+r_i\left(\frac{\partial^2\Phi_1}{\partial y^2}-\frac{\partial^2\Phi_2}{\partial y^2}\right)$$

$$=-\frac{\partial}{\partial y}\overline{\left(\frac{\partial\phi_1}{\partial x}(\nabla^2\phi_1+F_1\phi_2)\right)}. \tag{2.24a}$$

$$\mu\frac{\partial}{\partial T}(\nabla^2\Phi_2-F_2(\Phi_2-\Phi_1))+r_2\frac{\partial^2\Phi_2}{\partial y^2}+r_i\left(\frac{\partial^2\Phi_2}{\partial y^2}-\frac{\partial^2\Phi_1}{\partial y^2}\right)$$

$$=-\frac{\partial}{\partial y}\overline{\frac{\partial\phi_2}{\partial x}(\nabla^2\phi_2+F_2\phi_1)}-\frac{1}{\varepsilon}\frac{\partial}{\partial y}\overline{\left(\frac{\partial\phi_2}{\partial x}\eta_B\right)}. \tag{2.24b}$$

Since μ and the friction parameters are $O(\varepsilon^2)$ it is necessary to calculate the right-hand sides of (2.24a, b) to this order. Fortunately, it is not necessary to calculate $\phi_n^{(2)}$ explicitly as

$$-\frac{\partial}{\partial y}\overline{\left(\frac{\partial\phi_1}{\partial x}(\nabla^2\phi_1+F_1\phi_2)\right)}$$

$$=\varepsilon^2\frac{\partial}{\partial y}\overline{\left[\phi_1^{(0)}\frac{\partial}{\partial x}(\nabla^2\phi_1^{(2)}+F_1(\alpha\phi_1^{(2)}+\phi_2^{(2)}))\right]}, \qquad (2.25a)$$

$$-\frac{\partial}{\partial y}\overline{\left(\frac{\partial\phi_2}{\partial x}(\nabla^2\phi_2+F_2\phi_1)\right)}$$

$$=\varepsilon^2\frac{\partial}{\partial y}\overline{\left[\phi_2^{(0)}\frac{\partial}{\partial x}\left(\nabla^2\phi_2^{(2)}+F_2\left(\phi_1^{(2)}+\frac{1}{\alpha}\phi_2^{(2)}\right)\right)\right]}, \qquad (2.25b)$$

and we can substitute directly from (2.21a, b). Then, using (2.11), (2.16), (2.17) and the relation

$$\overline{\phi_2^{(0)2}}=\tfrac{1}{2}|A|^2\sin^2 ly,$$

we get

$$\frac{\mu}{\varepsilon^2}\frac{\partial}{\partial T}(\nabla^2\Phi_1-F_1(\Phi_1-\Phi_2))+\frac{r_1}{\varepsilon^2}\frac{\partial^2\Phi_1}{\partial y^2}+\frac{r_i}{\varepsilon^2}\left(\frac{\partial^2\Phi_1}{\partial y^2}-\frac{\partial^2\Phi_2}{\partial y^2}\right)$$

$$=\frac{l}{4U_1}\left[\frac{\mu}{\varepsilon^2}\frac{\alpha^2(\alpha+1)F_1^3}{F_2^2}\frac{d}{dT}|A|^2+\left(\frac{2r_1}{\varepsilon^2}\frac{a^2\alpha^2F_1^2}{F_2^2}\right.\right.$$

$$\left.\left.+\frac{2r_1}{\varepsilon^2}\frac{a^2\alpha F_1}{F_2}\left(\frac{\alpha F_1}{F_2}-1\right)\right)|A|^2\right]\sin 2ly, \qquad (2.26)$$

and

$$\frac{\mu}{\varepsilon^2}\frac{\partial}{\partial T}(\nabla^2\Phi_2-F_2(\Phi_2-\Phi_1))+\frac{r_2}{\varepsilon^2}\frac{\partial^2\Phi_2}{\partial y^2}+\frac{r_i}{\varepsilon^2}\left(\frac{\partial^2\Phi_2}{\partial y^2}-\frac{\partial^2\Phi_1}{\partial y^2}\right)$$

$$=-\frac{l}{4\alpha U_1}\left[\frac{\mu}{\varepsilon^2}\frac{(\alpha+1)}{\alpha}F_2\frac{d}{dT}|A|^2\right.$$

$$\left.+\left(\frac{2r_2}{\varepsilon^2}a^2+\frac{2r_i}{\varepsilon^2}a^2\left(1-\frac{\alpha F_1}{F_2}\right)\right)|A|^2\right]\sin 2ly. \qquad (2.27)$$

The three Eqs. (2.23), (2.26) and (2.27) determine the behaviour of the topographic wave amplitude A and the corrections to the zonal flow in each layer. As is the case for the barotropic flow on a β-plane the topography does not enter explicitly into the equation for the zonal flow, which changes only as a result of dissipation and secular variations in the wave.

3. INVISCID TOPOGRAPHIC INSTABILITY

In a frictionless flow, a steady solution to (2.23), (2.26) and (2.27) is

$$\Phi_1 = \Phi_2 = 0, \quad A \equiv A_0 = \frac{h_0}{\varepsilon\Delta} \frac{\alpha^2 F_2}{F_2^2 - \alpha^4 F_1^2}. \tag{3.1}$$

Note that, as required, $A = O(1)$ since $h_0 = O(\varepsilon^3)$ and $\Delta = O(\varepsilon^2)$. We assume, for the moment, that $\Delta \neq 0$ and that the undisturbed flow is not marginally stable ($\alpha^2 F_1 \neq F_2$). This solution represents a standing wave of height $O(h_0^{1/3})$, and when the momentum flux $\rho_0 h_2 U_2^2$ of the lower layer exceeds that of the upper layer $\rho_0 h_1 U_1^2$ it is a wave of elevation. If $\rho_0 h_2 U_2^2 < \rho_0 h_1 U_1^2$ it is a wave of depression.

We examine the stability of (3.1) to small perturbations by writing

$$A = A_0 + A', \quad \Phi_n = \Phi_n', \tag{3.2}$$

then writing $A' = A_r - iA_i$, where A_r and A_i are real, the linearized forms of (2.23), (2.26) and (2.27) are (on dropping dashes)

$$\frac{\partial^2 \Phi_1}{\partial y^2} - F_1(\Phi_1 - \Phi_2) = \frac{l}{2U_1} \frac{\alpha^2(\alpha+1)F_1^3}{F_2^2} A_0 A_r \sin 2ly \equiv C \sin 2ly, \tag{3.3}$$

$$\frac{\partial^2 \Phi_2}{\partial y^2} - F_2(\Phi_2 - \Phi_1) = -\frac{l}{2\alpha U_1} \frac{\alpha+1}{\alpha} F_2 A_0 A_r \sin 2ly \equiv D \sin 2ly, \tag{3.4}$$

$$\begin{aligned} &\frac{\mu}{\varepsilon^2}(\alpha+1)(\alpha^4 F_1^2 - F_2^2)\frac{dA}{dT} - \frac{\Delta}{\varepsilon^2} ikU_1(\alpha^4 F_1^2 - F_2^2)A \\ &\quad + 2A_0 ikl \int_0^1 \left[\alpha^4 F_1 \left(\frac{\partial^2 \Phi_1}{\partial y^2} + F_1 \Phi_2 \right) \right. \\ &\quad \left. - \alpha F_2 \left(\frac{\partial^2 \Phi_2}{\partial y^2} + F_2 \Phi_1 \right) + \alpha^5 F_1^2 \Phi_1 - F_2^2 \Phi_2 \right] \sin 2ly \, dy = 0. \end{aligned} \tag{3.5}$$

Define the variables

$$\xi = F_2\Phi_1 + F_1\Phi_2, \quad \chi = \Phi_1 - \Phi_2. \tag{3.6}$$

Then, using the boundary conditions (2.8), the solutions to (3.3) and (3.4) are

$$\xi = -\left(\frac{F_2C + F_1D}{4l^2}\right)(\sin 2ly - 2l(y - \tfrac{1}{2})), \tag{3.7}$$

$$\chi = -\left(\frac{C-D}{F_1 + F_2 + 4l^2}\right) \times\left(\sin 2ly - \frac{2l}{(F_1+F_2)^{1/2}} \frac{\sinh\{(F_1+F_2)^{1/2}(y-\frac{1}{2})\}}{\cosh[(F_1+F_2)^{1/2}/2]}\right). \tag{3.8}$$

The equation for $A(T)$ is then obtained by using (3.6)–(3.8) to determine Φ_1 and Φ_2 and substituting into the integrand in (3.5). In practice this turns out to be a complicated procedure and so we will outline the analysis schematically. After the substitutions have been made (3.5) may be written as

$$\frac{dA}{dT} - i\frac{\Delta}{\mu}\frac{kU_1}{1+\alpha}A + i\frac{\varepsilon^2}{\mu}kA_0^2A_rG(F_1, F_2, \alpha, l) = 0. \tag{3.9}$$

Since $G(F_1, F_2, \alpha, l)$ is real, we split (3.9) into real and imaginary parts

$$\frac{dA_r}{dT} + \frac{\Delta}{\mu}\frac{kU_1}{1+\alpha}A_i = 0, \tag{3.10}$$

$$\frac{dA_i}{dT} - \frac{\Delta}{\mu}\frac{kU_1}{1+\alpha}A_r + \frac{\varepsilon^2}{\mu}kA_0^2GA_r = 0. \tag{3.11}$$

Equations (3.10) and (3.11) admit solutions of the form $A = \hat{A}\, e^{\sigma T/\mu}$ where

$$\sigma^2 = \frac{k^2\Delta U_1}{1+\alpha}\left[\varepsilon^2 A_0^2 G - \frac{\Delta U_1}{1+\alpha}\right]. \tag{3.12}$$

Noting that $\alpha>0$, and substituting for A_0 from (3.1), we find that instability will occur if

$$(\Delta U_1)^3<\frac{h_0^2F_2^2(1+\alpha)\alpha^4U_1^2}{(\alpha^4F_1^2-F_2^2)^2}|G|. \tag{3.13}$$

The flow will exhibit either subreasonant or super-resonant instability depending on the sign of $G(F_1, F_2, \alpha, l)$, providing that the flow is sufficiently close to resonance.

For convenience we denote by δ the ratio of the layer depths h_1/h_2. Then writing $F_1=F$, we have $F_2=\delta F$. With this notation it is straightforward, but lengthy, to show that G, as defined by (3.9) and determined as described above, is given by

$$\alpha^2U_1G=\frac{[\delta(\alpha^8-\delta^4)+\alpha\delta^4(1+\delta)]4l^2F-16l^4\alpha(\alpha^7+\delta^4)-\delta F^2(\alpha^4-\delta^2)^2}{8\delta^2(\alpha^4-\delta^2)[F(1+\delta)+4l^2]}$$
$$-\frac{F(1+\alpha)(\alpha^4-\delta^2)}{4\delta(1+\delta)}-\frac{8l^4F^{1/2}(\alpha^4+\delta^3)^2\tanh[\frac{1}{2}F^{1/2}(1+\delta)^{1/2}]}{\delta^2(1+\delta)^{3/2}(\alpha^4-\delta^2)[F(1+\delta)+4l^2]^2}. \tag{3.14}$$

Note that when $\delta=\alpha^2$ the topographic wave is marginally stable to baroclinic instability [see (2.12)] and G is undefined. In this case it is necessary to rescale the time in the perturbation expansion.

First let us examine some special cases. Consider the case when the upper layer is shallow ($\delta\to 0$) but with the upper layer Froude number $F=O(1)$. Then it is readily seen from (3.14) that $G<0$ over the whole range of stable cross-stream wavenumbers $2\delta^{1/2}F<l^2<\infty$. Thus only subresonant instability is possible. When the upper layer is deep compared with the lower layer ($\delta\to\infty$) and with the lower layer Froude number $\delta F=O(1)$ then the situation is more complex. For $\alpha=O(1)$ (3.14) shows that

$$\alpha^2U_1G\to\frac{l^2}{2}[1-\alpha+16l^2(\delta F)^{-3/2}]. \tag{3.15}$$

Thus for $\alpha<1$ (the speed of the lower layer is less than that of the upper layer) the instability is always super-resonant. On the other hand, if $\alpha > 1 + 16l^2(\delta F)^{-3/2}$, then subresonant instability can occur. However, since $\alpha^2 > 2\delta^{1/2}F$ for baroclinic stability, subresonant instability will only occur at relatively large values of $\alpha = |U_2/U_1|$.

For the case of equal layer depths ($\delta = 1$) then consideration of (3.14) with $l^2 > 2F$ shows that $G > 0$ when $\alpha < 1$ and $G < 0$ when $\alpha > 1$. Thus the instability is super-resonant when the upper layer is the faster of the two layers and vice-versa. This last result agrees with the calculations of Pedlosky (1981) for two-layer flow on a β-plane. His analysis is restricted to equal layer depths and he only considers the case when the layer is stationary. Under these circumstances he found that super-resonant conditions were required for instability.

4. EQUILIBRIUM SOLUTIONS

Consider the case where the friction coefficients associated with the top and bottom Ekman layers are equal ($r_1 = r_2 = r$). We seek steady finite amplitude baroclinic wave solutions to (2.23), (2.26) and (2.27). Setting the time derivatives to zero these equations become

$$\frac{\partial^2 \Phi_1}{\partial y^2} + \gamma\left(\frac{\partial^2 \Phi_1}{\partial y^2} - \frac{\partial^2 \Phi_2}{\partial y^2}\right) = \frac{l}{2U_1} a^2 \left\{\frac{\alpha^2 F_1^2}{F_2^2} + \frac{\gamma \alpha F_1}{F_2}\left(\frac{\alpha F_1}{F_2} - 1\right)\right\} |A|^2 \sin 2ly, \tag{4.1}$$

$$\frac{\partial^2 \Phi_2}{\partial y^2} + \gamma\left(\frac{\partial^2 \Phi_2}{\partial y^2} - \frac{\partial^2 \Phi_1}{\partial y^2}\right) = -\frac{la^2}{2\alpha U_1}\left\{1 + \gamma\left(1 - \frac{\alpha F_1}{F_2}\right)\right\} |A|^2 \sin 2ly, \tag{4.2}$$

and

$$\begin{aligned}
&\frac{\Delta}{\varepsilon^2} ikU_1(\alpha^4 F_1^2 - F_2^2)A \\
&\quad - \left[\frac{r}{\varepsilon^2} a^2 \alpha(\alpha^2 F_1 - F_2) + \frac{r_i}{\varepsilon^2} a^2 \alpha(1 + \alpha^2)(F_2 - \alpha F_1)\right] A \\
&\quad - 2Aikl \int_0^1 \left[\alpha^4 F_1 \left(\frac{\partial^2 \Phi_1}{\partial y^2} + F_1 \Phi_2\right) - \alpha F_2 \left(\frac{\partial^2 \Phi_2}{\partial y^2} + F_2 \Phi_1\right)\right. \\
&\quad \left. + \alpha^5 F_1^2 \Phi_1 - F_2^2 \Phi_2\right] \sin 2ly \, dy = -\frac{ikU_1 h_0 \alpha^2 F_2}{\varepsilon^3},
\end{aligned} \tag{4.3}$$

where $\gamma = r_i/r$ is the ratio of the interface friction to the bottom friction. We write these three equations schematically as

$$\frac{\partial^2\Phi_1}{\partial y^2} + \gamma\left(\frac{\partial^2\Phi_1}{\partial y^2} - \frac{\partial^2\Phi_2}{\partial y^2}\right) = P|A|^2 \sin 2ly, \tag{4.4}$$

$$\frac{\partial^2\Phi_2}{\partial y^2} + \gamma\left(\frac{\partial^2\Phi_2}{\partial y^2} - \frac{\partial^2\Phi_1}{\partial y^2}\right) = Q|A|^2 \sin 2ly, \tag{4.5}$$

$$iRA - SA - iAI = iT \tag{4.6}$$

where the coefficients P, Q, R, S, T and I correspond in an obvious way to the terms in (4.1)–(4.3) and are all strictly real (assuming without loss of generality that h_0 is real).

The solutions to (4.4) and (4.5) subject to the boundary conditions (2.8) are

$$\Phi_1 + \Phi_2 = -\frac{P+Q}{4l^2}|A|^2\{\sin 2ly - 2l(y-\tfrac{1}{2})\}, \tag{4.7}$$

$$\Phi_1 - \Phi_2 = -\frac{P-Q}{4l^2(1+2\gamma)}|A|^2\{\sin 2ly - 2l(y-\tfrac{1}{2})\}. \tag{4.8}$$

From (4.6) and its complex conjugate we find

$$|A|^2 = -2(T/S)A_i, \tag{4.9}$$

and substituting in (4.7) and (4.8) we get that

$$\Phi_n = K_n(A_i/S)[\sin 2ly - 2l(y-\tfrac{1}{2})], \quad n=1,2, \tag{4.10}$$

where

$$K_1 = \frac{T}{l^2}\frac{[P(1+\gamma)+\gamma Q]}{1+2\gamma}, \quad K_2 = \frac{T}{l^2}\frac{[\gamma P + (1+\gamma)Q]}{1+2\gamma}. \tag{4.11}$$

Divide (4.6) into real and imaginary parts and substitute for Φ_n into the expression for I and we find

$$-RA_i - SA_r + IA_i = 0, \quad RA_r - SA_i - IA_r = T, \tag{4.12a, b}$$

where

$$I=[(\tfrac{3}{2}\alpha^5F_1^2+\tfrac{3}{2}\alpha F_2^2-2l^2\alpha^4F_1)K_1$$

$$+(\tfrac{3}{2}\alpha^4F_1^2+\tfrac{3}{2}F_2^2-2l^2\alpha F_2)K_2]\frac{A_i}{S}\equiv K_3A_i. \quad (4.13)$$

Solving (4.12a, b) and (4.13) gives a cubic for A_i:

$$K_3^2A_i^3-2K_3RA_i^2+(R^2+S^2)A_i+ST=0. \quad (4.14)$$

A more direct approach is to solve the equation as a quadratic for R which is proportional to Δ the departure from resonance. The solution is

$$R=K_3A_i\pm\left(-\frac{ST}{A_i}-S^2\right)^{1/2}. \quad (4.15)$$

Since $T>0$, we see immediately that for steady solutions to exist it is necessary that $S/A_i<0$. The correction to the zonal flow

$$\varepsilon U_n^{(1)}=-\partial\Phi_n/\partial y=2lK_n(A_i/S)\{1-\cos 2ly\}. \quad (4.16)$$

When there is no interfacial friction ($\gamma=0$) $K_1>0$, $K_2<0$, so that $U_1^{(1)}<0$ and $U_2^{(1)}>0$ and the velocity difference between the layers is decreased. When $\gamma=1$, it is readily seen from (4.11) that $K_1>0$ and $K_2>0$ for small δ, $K_1<0$ and $K_2<0$ for large δ and $K_1=0$ when $\delta=\delta_1=\frac{1}{4}\alpha[(1-2\alpha)+(4\alpha^2+28\alpha+1)^{1/2}]$ while $K_2=0$ when $\delta=\delta_2=\frac{1}{8}\alpha[(2-\alpha)+(\alpha^2+28\alpha+4)^{1/2}]$. Hence at small δ the lower layer velocity is reduced and the upper layer velocity is increased, and the opposite is true at large γ. In both cases the topographic drag decreases the baroclinic transport by the zonal flow in the steady state.

Define

$$\chi=\frac{A_iK_3^{2/3}}{(ST)^{1/3}}, \quad \eta=\frac{S^{4/3}}{(K_3T)^{2/3}}. \quad (4.17)$$

Then at resonance when $R=0$, (4.14) implies that

$$\chi^3+\eta\chi+1=0, \tag{4.18}$$

which has the *finite* solution $\chi=-1+\eta/3$ for small η (small friction). In addition define

$$\xi=-R/(K_3ST)^{1/3}. \tag{4.19}$$

Then (4.15) becomes

$$\xi=-\chi\pm(-\chi^{-1}-\eta)^{1/2}. \tag{4.20}$$

This equation has the same form as that found by Pedlosky (1981) for barotropic equilibrium on a β-plane, and so discussion of it here will be brief. The solution $\xi=\xi(\chi)$ of (4.20) is shown on Figure 1 for two values of the friction coefficient $\eta=0.1$ and $\eta=1$. For the smaller of the two frictional coefficients η the solution is multiple valued over a range of ξ, whilst at the larger of the two values the relationship $\xi=\xi(\chi)$ is single valued.

At small η when the solution is multivalued simple stability analysis shows that the uppermost and lowermost branches are stable whilst the intermediate branch is unstable. Hart (1979) has shown that for arbitrary initial conditions the solution will be captured by one of the stable branches of the equilibrium solutions. A solution on the uppermost branch will remain on the branch as ξ increases until $\xi=\xi_1$, at which point it will move back on to the uppermost branch. Thus the system will exhibit a marked hysteresis and may exhibit two different flows for the same values of the external parameters. On the uppermost branch the flow is characterised by a relatively large wave-amplitude and a weak mean flow whilst on the lowermost branch the wave-amplitude is small and the mean flow is stronger.

5. SUMMARY AND CONCLUSION

We have examined the flow of a two-layer flow on an f-plane over stationary topography with sinusoidal variation of height in the

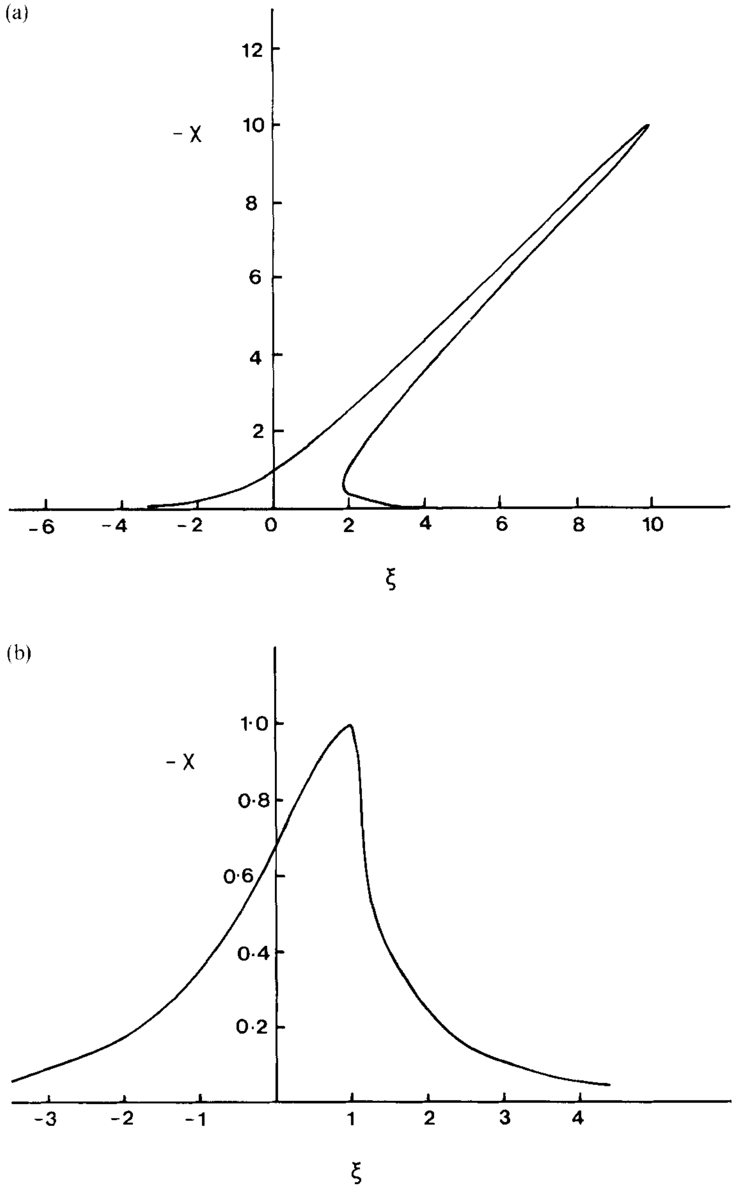

FIGURE 1 A measure of the wave amplitude plotted against departure from resonance for (a) small friction ($\eta = 0.1$) and (b) large friction ($\eta = 1$).

downstream direction. It is shown that when the two layers move in opposite directions a linear resonance exists producing a standing wave over the topography. The stability of the inviscid flow near this linear resonance is investigated and it is found that both subresonant and super-resonant instability is possible.

When the upper layer is shallow only subresonant instability is possible. On the other hand, when the depth of the upper layer is greater than that of the lower layer the instability is usually super-resonant although subresonant instability can occur when the speed of the lower layer is much larger than that of the upper layer. When the two layers have equal depths the instability is found to be super-resonant when the speed of the upper layer is the greater and vice-versa.

The equilibration of the instability at finite amplitude produced by friction at interfacial and top and bottom Ekman layers shows that the topographic drag reduces the transport by the mean flow. This reflects the fact that the form drag exerted by the topography reduced the potential energy of the flow and it is this energy source which drives the instability. It should be emphasised that this release of potential energy occurs when the flow in the absence of topography is baroclinically stable.

When the flow is marginally stable to baroclinic waves then the expansion procedure given above must be modified by rescaling the slow time $T=\mu t$ so that $\mu=O(\varepsilon)$ rather than $O(\varepsilon^2)$ as used in Section 3. The perturbation analysis can be carried through in a straightforward manner although the details will not be presented here. We find, as expected, that the influence of the topography is to stabilize the baroclinic waves (see Pedlosky, 1981).

The wave at the linear resonance is found to be finite in the presence of Ekman dissipation, and for small values of the dissipation the system exhibits multiple equilibria. For a given set of external parameters two stable equilibria exist, one characterised by a strong mean flow and small wave amplitude and the other by a relative large amplitude wave and a weaker mean flow. The latter has been suggested as a possible cause of atmospheric blocking. We have shown that similar dynamics prevail even when the scale of the motion is small and the variations in the Coriolis parameter with latitude can be neglected. This suggests that multiple equilibria may exist in ocean currents flowing over relatively small scale bottom topography.

I shall conclude with a few remarks on the feasibility of realising this flow in the laboratory. The fact that linear resonance is only achieved when the two layers move in opposite directions relative to the topography implies that it is not possible to drive the flow using differential rotation of a single lid. Provided the sealing problems could be overcome, the basic flow could be set up using an annulus with counter rotating top and bottom. Another possibility would be to pump the fluid in opposite directions in the two layers. A more worrying difficulty is that multiple equilibria are only obtained for small values of dissipation. On the laboratory scale Ekman dissipation is generally quite significant and it may be necessary to use a free upper surface and a relatively diffuse interface to eliminate interfacial Ekman layers.

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